ADM College for Women (Autonomous)

(Accredited with 'A' Grade by NAAC 4^{th} Cycle)

(Affiliated to Bharathidasan University, Thiruchirappalli)

Nagapattinam – 611 001

Department : Economics

Course Name : Mathematical Methods for Economic Analysis

Class : I MA

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Introduction

Mathematical Methods is an approach to economic analysis where mathematical symbols and theorems are used. Modern economics is analytical and mathematical in structure. Thus, the language of mathematics has deeply influenced the whole body of the science of economics. Every student of economics must possess a good proficiency in the fundamental methods of mathematical economics. One of the significant developments in Economics is the increased application of quantitative methods.

Syllabus

UNIT I: BASIC CONCEPTS

Constants, Variables, Parameters, Coefficient, Functions – Inverse, General and Specific Functions – Types of Functions - Linear and Non – linear. Uses of Mathematics in Economics. (Only Theory).

UNIT II: DIFFERENTIAL CALCULUS

Differentiation – Definition, sign of Derivatives, Rules of Differentiation – Basic rule, power rule, power of power rule, Addition and Subtraction rule and Quotient rule. Differentiation of Simple Functions Like $y = 5x^3 - 10x^2 + x - 100$. (No Logarithmic, Exponential and Trigonometric functions). Partial derivatives only for two variables, Higher order derivatives up to second order for the functions like Z =f (x, y) $=10x^3y^2 - 5x^2y + 6x - 11y + 1000$. Simple problems.

UNIT III: MATRIX ALGEBRA

Matrix – Definition – Types, Addition and Subtraction of 2 or more matrices, Scalar multiplication of a matrix, Multiplication of 2 matrices, (up to 3x3 order) Singular matrix, Non-Singular matrix, Uses of matrices-Simple problems.

UNIT IV: SOLVING SIMULTANEOUS EQUATIONS

Determinants – Definition, Minors and Co-factors of each element of a determinant (Simple problems…. No properties of determinants). Solving simultaneous equations using Cramer’s rule up to 3x3 order – Simple problems.

UNIT V: INPUT - OUTPUT MODEL

Leontief’s Input – Output model – Definition, Assumptions, Input Output Transaction Matrix, Closed and Open Input – Output models, Uses of Input – Output model, Limitations of Input – Output model (Only theory).

Unit I: BASIC CONCEPTS

Constants

A constant in math is a fixed value**.** It may be a number on its own or a letter that stands for a fixed number in an equation.

For example, in the equation "6x - 4 = 8," both 4 and 8 are constants because their values are fixed. In the equation "7x2 - 3x + 6," the number 6 is a constant, whereas 7 is the multiplying coefficient and "x" is referred to as the variable, not known until the equation is solved. The operator (such as "+" or "-") illustrates the function performed to satisfy or solve the equation.

Variables

Variables are usually letters or other symbols that represent unknown numbers or values.

Examples of Variable:

The following are examples of algebraic expressions and equations containing variables.

3x + 2 = 20, the variable here is x

3y + 10 = 31, the variable here is y

a2 + b2, the variables here are a and b

In Mathematics, we usually deal with two types of quantities-Variable quantities (variables) and Constant quantities (or constants). If the value of a quantity remains unaltered under different situations, it is called a constant. On the contrary, if the value of a quantity changes under different situations, it is called a variable. For example 4, 2.718, 22/7 etc. are constants while the speed of a train, the demand for a commodity, population of a town etc. are variables.

In problems relating to two or more variables, it is seen that the value of variable changes with the change in the value (or values) of the related variable (or variables). Suppose a train running at a uniform speed of v km./h. travels a distance of d km. in t hours. Obviously, if t remains unchanged then v increases or decreases according to as d increases or decreases. But if d remains unchanged, then v decreases or increases according to as ‘t’ increases or decreases.

This shows that the change in the value of a variable may be accompanied differently with the change in the values of related variables. Such relationship with regards to the change in the value of a variable when the value of the related variables change is termed as a variation.

Parameter

A Parameter is a quantity that influences the output or behavior of a mathematical object but is viewed as being held constant. Parameters are closely related to variable each experiment and only change between experiments.

For example, a function might a generic quadratic function as $F(x) = ax^2+bx+c$. Here the variable x is regarded as the input to the function

The symbols a, b, c are parameters that determine the behavior of the function f.

For each value of the parameter, we get a different function.

Coefficient

A numerical or constant quantity placed before and multiplying the variable in an algebraic expression

Ex: $6x^3$ (6 is a coefficient)

Number or other known factor (usually a constant) by which another number or factor (Usually a variable) is multiplied.

For example, in the equation $ax_2+bx+c=0$

a is the coefficient of x_2

b is the coefficient of x

Variable, Constant, coefficient and Parameters

A variable is something whose magnitude can change i.e something that can take on different values. Variable frequently used in economics include price, profit, revenue, cost, national income, consumption, investment, imports, and exports. Since each variable can assume various values, it must be represented by a symbol instead of a specific number. For example, we may represent price by P, profit by Π, revenue by R, cost by C, national income by Y, and so forth. When we write P =3, or C = 18, however, we are "freezing" these variables at specific values.

Properly constructed, an economic model can be solved to give us the solution values of a certain set of variables, such as the market- clearing level of price, or the profit maximizing level of output. Such variable, whose solution values we seek from the model, as known as endogenous variable (originated from within). However, the model may also contain variables which are assumed to be determined by forces external to the model and whose magnitudes are accepted as given data only; such variable are called exogenous variable (originating from without). It should be noted that a variable that is endogenous to one model may very well be exogenous to another. In an analysis of the market determination of wheat price (p), for instance, the variable P should definitely by endogenous; but in the frame work of a theory of consumer expenditure, p would become instead a datum to the individual consumer, and must therefore be considered exogenous

Variable frequently appear in combination with fixed numbers or constants, such as in the expressions, 7 P or 0.5 R

A constant is a magnitude that does not change and is therefore the antithesis of a variable. When a constant is joined to a variable; it is often referred to as the coefficient of that variable. However, a coefficient may be symbolic rather than numerical.

As a matter of convention, parametric constants are normally represented by the symbols a,b,c, or their counterpart in the Greek alphabet α , β and λ. But other symbols naturally are also permissible.

Equation and Identities

Variable may exist independently, but they do not really become interesting until they are related to one another by equations or by inequalities. In economic equations, economist may distinguish between three types of equation: definitional equations, behavioral equations and conditional equations.

Function

A single constant or a variable is a function. If the function contains only a constant then it is referred to as a constant function. If constants and variables are related by means of a mathematical operator (+) or (-) or (*) or (/) it is referred to as a function**.**

Inverse Function

Consider the function y =f(x) . We can writef^{-1} (y) = x. It is called Inverse function.

Ex: $y = x^2 = f(X)$

$f^{-1}(x) = y^{-1} \quad = 1/y \quad = 1/x^2$

Linear Functions:

- Have graphs that are straight lines.
- The Rate of Change between any two points is constant.
- Variable terms can only have an exponent of one.

A linear function is a function which forms a straight line in a graph. It is generally a polynomial function whose degree is utmost 1 or 0. Although the linear functions are also represented in terms of calculus as well as linear algebra. The only difference is the function notation. Knowing an ordered pair written in function notation is necessary too. f(a) is called a function, where a is an independent variable in which the function is dependent.

Linear Function Graph has a straight line whose expression or formula is given by;

$$y = f(x) = px + q$$

It has one independent and one dependent variable. The independent variable is x and the dependent one is y. P is the constant term or the y-intercept and is also the value of the dependent variable. When x = 0, q is the coefficient of the independent variable known as slope which gives the rate of change of the dependent variable.

Nonlinear Function

- A function which is not linear is called nonlinear function. In other words, a function which does not form a straight line in a graph. The examples of such functions are exponential function, parabolic function, inverse functions, quadratic function, etc. All these functions do not satisfy the linear equation $y = m x + c$. The expression for all these functions is different.
- Have graphs that are NOT straight lines.
- The Rate of Change between any two points is NOT constant.
- Variable terms will have exponents that are numbers other than one.

We look at different types of nonlinear functions, including quadratic functions, polynomials and rational, exponential and logarithmic functions, as well as some applications such as growth and decay and financial functions.

Properties of Functions

A function is a rule which assigns to each element in one set one and only element from another set. A non-linear function is not a straight line. A function be described in different ways, such as using a set diagram, table or by equation. The domain is the set of all possible values of the independent variable of a function x; the range is the set of all possible values of the dependent variable of a function $y = f(x)$.

Two special types of functions are discussed, including:

- even function: $f(-x) = f(x)$, a function symmetric about the y-axis,
- odd function: $f(-x) = -f(x)$, a function symmetric about the origin.

Type of functions:

Demand function

Demand function express the relationship between the price of the commodity (independent variable) and quantity of the commodity demanded (dependent variable). It indicate how much quantity of a commodity will be purchased at its different prices. Hence, d_x represent the quantity demanded of a commodity and p_x is the price of that commodity. Then,

Demand function $\mathbf{d_x = f(p_x)}$

The basic determinants of demand function

$$Q_x = f(P_x, P_r, Y, T, W, E)$$

Here, Qx: quantity demanded of a commodity X

Px: price of commodity X,

Pr: price of related good,

Y: consumer's income,

T: Consumer/s tastes and preferences,

W: Consumer's wealth,

E: Consumer's expectations.

Supply function

Supply function express the relationship between the price of the commodity (independent variable) and quantity of the commodity supplied (dependent variable). It indicates how much quantity of a commodity that the seller offers at the different prices. Hence, S_x represent the quantity supplied of a commodity and p_x is the price of that commodity. Then,

Supply function = $\mathbf{f(S_x\ p_x)}$

The basic determinants of supply function

$$Q_s = f(G_f, P, I, T, P_r, E, G_p)$$

Here,Qs: quantity supplied, Gf: Goal of the firm, P: Product's own price, I: Prices of inputs,

T: Technology,

Pr: Prices of related goods,

E: Expectation of producer's,

Gp: government policy.

Utility function

People demand goods because they satisfy the wants of the people. The utility means wants satisfying power of a commodity. It is also defined as property of the commodity which satisfies the wants of the consumers. Utility is a subjective entity and resides in the minds of men. Being subjective it varies with different persons, that is, different persons derive different amounts of utility from a given good. Thus, the utility function shows the relation between utility derived from the quantity of different commodity consumed. A utility function for a consumer consuming different goods may be represented:

$$U = f(X_1, X_2, X_3 \ldots\ldots\ldots)$$

Example: For the utility unction of two commodities

$U = f(x_1 - 2)^2 (x_2 + 1)^3$, find the marginal utility of x_1 and x_2.

$= \frac{\partial u}{\partial x_1}$ **$2(x_1 - 2)(x_2 + 1)^3$** is the MU function of the first commodity,

$= \frac{\partial u}{\partial x_2}$ **$3(x_2 + 1)^2 (x_1 - 2)^2$** is the MU function of the second commodity

Consumption Function

The consumption function or propensity to consume denotes the relationship that exists between income and consumption. In other words, as income increases, consumers will spend part but not all of the increase, choosing instead to save some part of it. Therefore, the total increase in income will be accounted for by the sum of the increase in consumption expenditure and the increase in personal saving. This law is known as propensity to consume or consumption function. Keynes contention is that consumption expenditure is a function of absolute current income, ie:C = f (Yt)

The linear consumption function can be expressed as:

$$C = C_0 + b\,Y_d$$

Where, C_0 is the autonomous consumption, b is the marginal propensity to consume and Yd is the level of income.

Given the consumption function **C = 40 + 0.80Yd**, autonomous consumption function is 40 and marginal propensity to consume is **0.80**.

Production function

Production function is a transformation of physical inputs in to physical out puts. The output is thus a function of inputs. The functional relationship between physical inputs and physical output of a firm is known as production function. Algebraically, production function can be written as: $Q = f(a,b,c,d,\ldots..)$

Where, Q stands for the quantity of output, a,b,c,d, etc.; stands for the quantitative factors. This function shows that the quantity (q) of output produced depends upon the quantities, a, b, c, d of the factors A, B, C, D respectively.

The general mathematical form of Production function is:

$$Q = f(L,K,R,S,v,e)$$

Where Q stands for the quantity of output, L is the labour, K is capital, R is raw material, S is the Land, v is the return to scale and e is efficiency parameters.

Example: Suppose the production function of a firm is given by: $Q = 0.6X + 0.2Y$

Where, Q = Output, X and Y are inputs.

Cost Function

Cost function derived functions. They are derived from production function, which describe the available efficient methods of production at any one time. Economic theory distinguishes between short run costs and long run costs. Short run costs are the costs over a period during which some factors of production (usually capital equipments and management) are fixed. The long run costs over a period long enough to permit the change of all factors of production. In the long run all factors become variable. If x is the quantity produced by a firm at a total cost C, we write for cost function as:

$$\mathbf{C = f(x)}$$

It means that cost depends upon the quantity produced.

Revenue Function

If R is the total revenue of a firm, X is the quantity demanded or sold and P is the price per unit of output, we write the revenue function. Revenue function expresses revenue earned as a function of the price of good and quantity of goods sold. The revenue function is usually taken to be linear.

$\mathbf{R = P \times X}$

Where R = revenue, P = price, X = quantity

If there are n products and $P_1, P_2\ldots..P_n$ are the prices and $X_1, X_2\ldots\ldots X_n$ units of these products are sold then

Profit Function

Profit function as the difference between the total revenue and the total cost. If x is the quantity produced by a firm, R is the total revenue and C being the total cost then profit (π). $\boldsymbol{\pi}$ **= TR – TC**

Saving Function

The saving function is defined as the part of disposable income which is not spending on consumption. The relationship between disposable income and saving is called the saving function. The saving function can be written as: **S = f(Y)**

Where, S is the saving and Y is the income.

In mathematically the saving function is: S = c + bY

Where S is the saving, c is the intercept and b is the slope of the saving function.

Example: Suppose a saving function is S = 30 + 0.4 Y

Investment function

The investment function shows the functional relation between investment and the rate if interest or income. So, the investment function

I = f (i) Where, I is the investment and is the rate of interest In other way, the investment function

I = f (Y_{t+1} - Y_t), Shows that I is the investment, Y_{t+1} is future income, Yt is the present level of income. Investment is the dependent variable; the change in income is the independent variable.

Linear

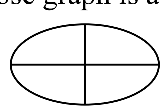
Linear function is those whose graph is a straight line. The linear function has the following form Y= mx+c

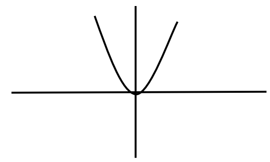
The linear function is popular in economics. The power of x and y is 1

Non-Linear

Non-Linear functions are those whose graph is a curve.

Ex: $x^2+y^2=r^2$ -Circle





Y=4ax - Parabola

The power of x or y or both are more than one. This type of equations are called non-linear equations

Even Function

$Y = f(x) = x^2$

➔ $f(-x) = (-x)^2 = x^2$

➔ $F(x) - (x)^2 = x^2$

Odd Function

$Y = f(x) = x^3$

➔ $f(-x) = (-x)^3 = x^3$

➔ $F(x) - (x)^3 = x^3$

Algebraic Function

The Function is of the form

$F(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots\ldots + a_n$

x= Variable $a_0, a_1, a_2 \ldots\ldots\ldots a_n$ Constants

Uses of Mathematics in Economics

Mathematical Economics is not a distinct branch of economics in the sense that public finance or international trade is. Rather, it is an approach to Economic analysis, in which the Economist makes use of mathematical symbols in the statement of the problem and also drawn up on known mathematical theorem to aid in reasoning. Mathematical economics insofar as geometrical methods are frequently utilized to derive theoretical results. Mathematical economics is reserved to describe cases employing mathematical techniques beyond simple geometry, such as matrix algebra, differential and integral calculus, differential equations, difference equations etc….

It is argued that mathematics allows economist to form meaningful, testable propositions about wide- range and complex subjects which could less easily be expressed informally. Further, the language of mathematics allows Economists to make specific, positive claims about controversial subjects that would be impossible without mathematics. Much of Economics theory is currently presented in terms of mathematical Economic models, a set of stylized and simplified mathematical relationship asserted to clarify assumptions and implications.

Mathematical economics is a model of economics that utilizes math principles and methods to create economic theories and to investigate economic quandaries. Mathematics permits economists to conduct quantifiable tests and create models to predict future economic activity.

Mathematical economics is a form of economics that relies on quantitative methods to describe economic phenomena. Although the discipline of economics is heavily influenced by the bias of the researcher, mathematics allows economists to explain observable phenomenon and provides the backbone for theoretical interpretation. The merge of statistical methods, mathematics and economic principles has created an entirely new branch of economics called Econometrics.

Mathematical economics is the application of mathematical methods to represent economic theories and analyze problems posed in economics. It allows formulation and derivation of key relationships in a theory with clarity, generality, rigor, and simplicity. By convention, the applied methods refer to those beyond simple geometry, such as differential and integral calculus, difference and differential equations, matrix algebra, and mathematical programming and other computational methods.

Mathematics allows economists to form meaningful, testable propositions about many wide-ranging and complex subjects which could not be adequately expressed informally. Further, the language of mathematics allows economists to make clear, specific, positive claims about controversial or contentious subjects that would be impossible without mathematics. Much of economic theory is currently presented in terms of mathematical economic models, a set of stylized and simplified mathematical relationships that clarify assumptions and implications.

Broad applications include:

1. Optimization problems as to goal equilibrium, whether of a household, business firm, or policy maker.
2. static (or equilibrium) analysis in which the economic unit (such as a household) or economic system (such as a market or the economy) is modeled as not changing
3. Comparative statics as to a change from one equilibrium to another induced by a change in one or more factors. dynamic analysis, tracing changes in an economic system over time, for example from economic growth
4. Formal economic modeling began in the 19th century with the use of differential calculus to represent and explain economic behavior, such as utility maximization, an early economic application of mathematical optimization. Economics became more

mathematical as a discipline throughout the first half of the 20th century, but introduction of new and generalized techniques in the period around the Second World War, as in game theory, would greatly broaden the use of mathematical formulations in economics.

UNIT - II
DIFFERENTIAL CALCULUS

Differentiation

Differentiation is the process of finding derivative of a function is called differentiation. If x and y are two variables, then rate of change of x with respect to y is the derivative.

We denote small increase in the consumption x1x2 by.(Δx) . This changes total utility by small amount y1y2 =.(Δy) therefore, the change in utility per unit will be

$$\frac{y_1y_2}{x_1x_2} = \frac{(\Delta y)}{(\Delta x)} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Since y=f(x) and y+Δy = f(x+Δx)

This fact is written as $\lim_{\Delta x \to 0} \frac{(\Delta y)}{\Delta x} = \frac{dy}{dx}$

The example of differentiation is velocity which is equal to rate of change of displacement with respect to time. Another example is acceleration which is equal to rate of change of velocity with respect to time.

Function	**Derivative of the function**
(i) h(x) = c.f(x) where c is any real constant (Scalar multiple of a function)	$\frac{d}{dx}\{h(x)\} = c.\frac{d}{dx}\{f(x)\}$
(ii) h(x) = f(x) ± g(x) (Sum/Difference of function)	$\frac{d}{dx}\{h(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}$
(iii) h(x) = f(x). g(x) (Product of functions)	$\frac{d}{dx}\{h(x)\} = f(x)\frac{d}{dx}\{g(x)\}+g(x)\frac{d}{dx}\{f(x)\}$
(iv) $h(x) = \frac{f(x)}{g(x)}$ (Quotient of function)	$\frac{d}{dx}\{h(x)\} = \frac{g(x)\frac{d}{dx}\{f(x)\}-f(x)\frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$
(v) h(x) = f{g(x)}	$\frac{d}{dx}\{h(x)\} = \frac{d}{dz}f(z).\frac{dz}{dx}$, where z = g(x)

Rules of Differentiation

Relationship between x and y, viz, $y = 16x^2$, but the relationship between x and y can take any form. Rules of differentiation are given below to deal with different forms of functions:

Rule 1: Functions in power terms (ax^n).

Let us suppose that the functional relationship between x and y is of the form: $y = ax^n$, where a is some constant term.

In this case $dy/dx = anx^{n-1}$

Thus, our first rule of Differentiation about power terms is as under:

If $y = ax^n$, $dy/dx = anx^{n-1}$

Rule 2: Differentiation of a constant.

Suppose the functional relationship is of the form:

$y = C$ (C being constant)

We can write this as $y = Cx^o$ (Since $x^o = 1$)

Applying Rule 1, $dy/dx = C(0)(x)^{0-1}$ $dy/dx=0$

If $y=C$; then $dy/dx= 0$;

Rule 3: Differentiation of sum and difference.

Let $y = 7x^3 + 5x^5 - 3x^6 + 8$.

We obtain dy/dx by differentiating each term separately by applying Rules 1 and 2.

$dy/dx = d(7x^3)/dx + d(5x^5)/dx - d(3x^6)/dx + d(8)/dx$

$$= \frac{d(7x^3)}{dx} + \frac{d(5x^5)}{dx} - \frac{d(3x^6)}{dx} + \frac{d(8)}{dx} \qquad = 21x^2 + 25x^4 - 18x^5$$

Rule 4: Differentiation of a product.

Supposing that y is given by the product of two expressions:

$$y = (4x^2 + 2x)(8x^3 + 3x^2)$$

We can differentiate this relationship by finding the product of the two terms and then differentiating term by term on the basis of above rules. But there is an easier method which we describe below.

Let us suppose $4x^2 + 2x = U$ and $8x^3 + 3x^2 = V$

So that, y = UV (U is first term; V is second term)

Then dy/dx can be obtained by the following rule:

$$\boxed{\text{If } y=UV;\text{ then } \frac{dy}{dx} = V.\frac{dU}{dx} + U.\frac{dV}{dx}}$$

Since $V = 8x^3 + 3x^2$

$\therefore\ dV/dx = 24x^2 + 6x$

and $U = 4x^2 + 2x$

$dU/dx = 8x+2$

$dy/dx = V.\ dU/dx + U.\ dV/dx$

$= (8x^3 + 3x^2)(8x+2) + (4x^2 + 2x)(24x^2 + 6x)$

Rule 5: Differentiation of a quotient

Suppose that y is expressed as a quotient of two functions of x,

That is: $y = \dfrac{8x^8 + 6x^2 - 2x}{3x^2 - 5x}$

Then, dy/dx can be found from the following rule:

$$\text{If } y = U/V;\text{ then } dy/dx = \frac{V.\left(\frac{dU}{dx}\right) - U.\left(\frac{dV}{dx}\right)}{V^2}$$

The same result will be obtained by applying product rule is

$Y = UV^{-1} = (8x^8 + 6x^2 - 2x)(3x^2 - 5x)^{-1}$

$U = 8x^8 + 6x^2 - 2x$

$dU/dx = 64x^7 + 12x - 2$

$V = 3x^2 - 5x$

$dV/dx = 6x+5$

$dx/dy = (3x^2 - 5x)(64x^7 + 12x - 2) - (8x^8 + 6x^2 - 2x)(6x+5)/(3x^2 - 5x)^{-1}$

Rule: 6 Differentiation of a function of a function (Chain Rule)

Let us suppose that consumption is not related to total utility directly but indirectly via income (I).

And our problem is what will be the change in total utility if consumption changes by one unit. Here we have to apply the rule for differentiation of a function of a function.

Assume that y =I/2 and I = x/5

Take the expression I =x/5 here differentiating I (Which represents change in I due to a small change in x)

dI/dx = 1/5

also, when y = I/2 then dy/dI (Which represents change in y due to a small change in I) =1/2
Thus, now we have two derivates, dI/dx = 1/5 and dy/dI =1/2

But we need the change in y with small change in x,i.e., we need dy/dx. The result can be obtained by the following Rule:

dy/dx = dy/dI. dI/dx = 1/2.1/5 = 1/10

This particular way of getting the desired derivative in case of a function of a function is known as chain Rule.

Rule 7: Differentiate of implicit function

If y and x are related in a form such as $y = 2x/x^2+1$, then y is said to be an explicit function of x; because the R.H.S. contains expression in x alone (no-y) and L.H.S. contains only y. In such cases y is expressed explicitly in terms of x.

An equation: $x^3y + y - 2x = 0$, does not express y in terms of x directly, we say that the given relation is an implicit one.

In such case we differentiate term by term with respect to (w.r.t) x in them following way: Given implicit relation: $x^3y + y -2x = 0$. dy/dx =2x+3/2y+5

The derivative of constant function is zero.
For example, if f(x) = 8, then f'(x) = 0.

Q1: **Differentiate** $\mathbf{y = x^n}$

$$d(y)/dx = d(x^n)/dx = nx^{n-1}$$

Q2: **Differentiate y = x⁴**

$dy/dx = d(x^4)/dx$
$= 4x^3$

Q3: **Differentiate $y = x^{11}$**

$dy/dx = d(x^{11})/dx$
$=11x^{10}$

Q4: **Differentiate $y = x^{7/3}$**

$dy/dx = d(x^{7/3-1})/dx$
$=7/3\ \mathbf{x^{4/3}}$

Q5: Differentiate $y = x^{11/2}$

$dy/dx = d(x^{11/2-1})/dx$
$=11/2\ \mathbf{x^{9/2}}$

Q6: Differentiate y = 5

$dy/dx = d(5)/dx$
$=0$

Q7: Differentiate $\mathbf{y=\sqrt{x}}$

$\mathbf{y = x^{1/2}}$
$dy/dx =1/2\ x^{1/2-1}$
$=1/2\ x^{-1/2}$
$= \frac{1}{2x^{1/2}}$
$= \frac{\mathbf{1}}{2\sqrt{x}}$

Q8: Differentiate

$y = x^3-x^2+5$
$dy/dx = d(x^3)/dx + d(x^2)/dx + d(5)/dx$

$= 3x^2 + 2x+0$

Q9**: Differentiate**

Given: $f(x) = 6x^3-9x+4$
On differentiating both the sides w.r.t x, we get;
$f'(x) = (3)(6)x^2 – 9$
$f'(x) = 18x^2 – 9$

Q10: Differentiate y = x

$dy/dx = 1x^{1-1}$
$=1x^0$
$=1X1=1$

Q11: Differentiate y = 5x

$dy/dx = d(5x)/dx$
$= 5d(x)/dx$
$=5(1)=5$

Q12: Differentiate $y = x(3x^3 - 9)$

Given, $y = x(3x^3 - 9)$

$y = 3x^3 - 9x$

On differentiating both the sides we get,

$dy/dx = 9x^2 - 9$

Q13: $y = (x^3)(x^2+2)$

$u = x^3$ $v = x^2+2$

$u' = 3x^2$ $v' = 2x+0 = 2x$

$d(uv) = uv' + vu'$

$dy/dx = d(uv)/dx$

$= d[(x^3)(x^2+2)]/dx$

$= (x^3)[2x] + (x^2+2)[3x^2]$

$= 2x^4+3x^4+6x^2$

$= 5x^4+6x^2$

Q14: y=x2+2x / x3 = u/v

$d(u/v)/dx = vu' - uv'/v^2$

$u = x^2+2x$ $v = x^3$

$u' = 2x+2$ $v' = 3x^2$

$dy/dx = d(u/v)/dx = d(x^2+2x/x^3) / dx$

$= (x^3)[2x+2] - [x^2+2x][3x^2] / (x^3)^2$

$= 2x^4+2x^3-[3x^4+6x^3] / x^6$

$= x^4 - 4x^3 / x^6$

Partial Derivatives:

Let's talk about functions of two variables here. You should be used to the notation y = f (x) for a function of one variable, and that the graph of y = f (x) is a curve. For functions of two variables the notation simply becomes, z = f (x, y).

Where the two independent variables are x and y, while z is the dependent variable. The graph of something like z = f (x, y) is a surface in three-dimensional space.

Since z = f (x, y) is a function of two variables, if we want to differentiate, we have to decide whether we are differentiating with respect to x or with respect to y (the answers are different). A special notation is used. We use the symbol ∂ instead of d and introduce the partial derivatives of z, which are:

$\frac{\partial z}{\partial x}$ is read as "partial derivative of z (or f) with respect to x", and means differentiate with respect to x holding y constant

$\frac{\partial z}{\partial y}$ means differentiate with respect to y holding x constant

Another common notation is the subscript notation:

$$zx \text{ means } \frac{\partial z}{\partial x}$$

$$zy \text{ means } \frac{\partial z}{\partial y}$$

EXAMPLE

1. $U = 5x-6y+8$
   $\partial U/\partial x = f_x = 5- 0+0 = 5$
   $\partial U/\partial y = f_x = 0- 6+0 = -6$

2. $Z= ax+by+c$
   $\partial Z/\partial x = a$
   $\partial Z/\partial y = b$

3. $Z= 4x^2+4xy+y^2$
   $\partial Z/\partial x = 8x+4y$
   $\partial Z/\partial y = 4x+2y$

4. $z = x^3 y^5 - 10x^2 y + 40x$

$\partial z/\partial x = 3x^2 y^5 - 20xy + 40$

$\partial z/\partial y = x^3(5y^4) - 10x^2(1) + 0$

$\partial z/\partial y = 5x^3 y^4 - 10x^2$

5. $z = 3x^2 y - 5x^4 y^2 + 10$

$\partial z/\partial x = 3y(2x) - 5y^2(4x^3) + 0$

$\partial z/\partial x = 6xy - 20x^3 y^2$

$\partial z/\partial y = 3x^2(1) - 5x^4(2y) + 0$

$\partial z/\partial y = 3x^2 - 10x^4 y$

6. Find $\partial^2 z/\partial x^2$,

$z = 3x^3 y + y^6 - 4y^2 + 3x^2$

$\partial^2 z/(\partial x^2) = \partial(\partial z/\partial x)/\partial x \rightarrow$

$\partial z/\partial x = 3y(3x^2) + 0 - 0 + 6x$

$= 9x^2 y + 6x$

$\partial(\partial z/\partial x)/\partial x = \partial(9x^2 y + 6x)/\partial x$

$= 9y(2x) + 6$

$= \partial^2 z/\partial x = 18xy + 6$

$\partial^2 z/\partial y\partial x = \partial(\partial z/\partial x)/\partial y$

$= \partial(9x^2 y + 6x)/\partial y$

$= \mathbf{9x^2(1) + 0}$

$\partial^2 z/\partial x^2 = \partial(\partial z/\partial x)/\partial x$

$\partial^2 z/(\partial y^2) = \partial(\partial z/\partial y)/\partial y$

$\partial^2 z/\partial x\partial y = \partial(\partial z/\partial y)/\partial x$

$\partial^2 z/\partial y\partial x = \partial(\partial z/\partial x)/\partial y$

Higher Order Derivatives:

Because the derivative of a function y = f(x) is itself a function y′ = f′(x), you can take the derivative of f′(x), which is generally referred to as the second derivative of f(x) and written f''(x) or f 2(x). This differentiation process can be continued to find the third, fourth, and successive derivatives of f(x), which are called higher order derivatives of f(x). Because the "prime" notation for derivatives would eventually become somewhat messy, it is preferable to use the numerical notation f (n)(x) = y (n) to denote the nth derivative of f(x).

Find the first and second derivatives

F(x)= $5x^4-3x^3+7x^2-9x+2$

F'(x) =$20x^3-9x^2+14x-9$

F''(x) =$60\ x^2+18x+14$

F'''(x) = 120x-18

Sign of Derivatives:

- When the sign of the derivative is positive, the graph is increasing. The sign of the derivative is positive for all values of x > 0.
- When the sign of the derivative is negative, the graph is decreasing. The sign of the derivative is negative for all values of x < 0.
- The sign of the derivative for the function is equal zero at the minimum of the function. The derivative is zero when x = 0
- The function is increasing when the sign of the derivative is positive. The derivative is positive for all values of x.
- The derivative of the function is always positive. There are no x values that yield a negative derivative.
- The sign of the derivative for the function never equals zero.
- The derivative of the function is always negative. There are no x values that yield a positive derivative.
- The function is decreasing when the sign of the derivative is negative. The derivative is negative for all values of x.

UNIT- 3 MATRIX

What is a Matrix?

A Matrix is a defined as a rectangular array of elements arranged in rows and columns.

Its general form is

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots a_{2n} \\ a_{m1} & a_{m2} \ldots\ldots a_{mn} \end{bmatrix}$$

This is a matrix of m rows and n columns. It donoted by (a_{ij}) and is read m by n.

The size or dimension of a matrix is defined by the number of rows and columns it contains.

Numbers written in such a particular form or rows and columns and enclosed by [ ]→ square brackets or ( ) → large brackets or || || → double bars are called a matrix.

Order of matrix:

If the matrix is not just an aggregate of numbers; because in a given matrix each element has its assigned position in a particular row and columns.

It is to be remembered that

Matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix}$ is not equal to $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \end{pmatrix}$

4. Matrix A = Matrix B of they have the same order and each element of A is equal to the corresponding element of B.
5. If A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and B = $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

then A =B if and only if $a_{11}=b_{11}, a_{12}=b_{12} \ldots\ldots a_{ij} = b_{ij}$

3. A Matrix of m rows and n columns is an m X n matrix.
4. If m=n, the matrix is a square matrix of order n.
5. If the matrix consists of only one column as $\begin{pmatrix} a1 \\ a2 \\ a3 \end{pmatrix}$, it is a column matrix or column vector.
6. If a matrix consists of only one row as $[a1\ b1\ c1]$
7. A matrix is said to be a zero or Null matrix if and only if each of its elements is zero.

Addition and Subtraction of Matrices

A = $\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}$ and B = $\begin{matrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{matrix}$

There are basically two criteria which define the addition of matrix. They are as follows:

1. Consider two matrices A & B. These matrices can be added if (if and only if) the orders of matrix are equal i.e., the two matrices have the same number of rows and columns. For example, say matrix A is of the order 3X 4, then the matrix B can be added to matrix A if the order of B is also 3X 4.
2. The addition of matrices is not defined for matrices of different sizes.

Properties of Matrix Addition

Assume that, A, B and C be three m x n matrices,

The following properties holds true for the matrix addition operation.

A + B = B + A (commutative property)

A + (B + C) = (A + B) + C (associative property)

For any m x n matrix, there is an identity element)

A + 0 = A (where 0 is an additive identity)

For any m x n matrix A there is an m x n matrix B

A + B = O (B is an additive inverse of A, which is equal to -A)

Ex.1 If $A=\begin{bmatrix} 2 & 0 \\ -5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 6 \\ 4 & 1 \end{bmatrix}$

Find $A+B = A = \begin{bmatrix} 2 & 0 \\ -5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 6 \\ 4 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2+-3 & 0+6 \\ -5+6 & 6+1 \end{bmatrix} = \begin{bmatrix} -1 & 6 \\ -1 & 7 \end{bmatrix}$$

Ex.2

If $A=\begin{bmatrix} 2 & 0 \\ -5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 6 \\ 4 & 1 \end{bmatrix}$

Find $A-B = A = \begin{bmatrix} 2 & 0 \\ -5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 6 \\ 4 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2- -3 & 0-6 \\ -5-6 & 6-1 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ -11 & 5 \end{bmatrix}$$

Matrix Multiplication

There are two types of multiplication for matrices: scalar multiplication and matrix multiplication. Scalar multiplication is easy. You just take a regular number (called a "scalar") and multiply it on every entry in the matrix.

For the following Matrix A, find 2A and -1 A

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

To do the first scalar multiplication to find 2A, I just multiply a 2 on every entry in the matrix:

$$2\,A = 2\,.\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2.1 & 2.2 \\ 2.3 & 2.4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

The other scalar multiplication, to find –1A, works the same way:

$$-1\,A = -1\;.\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -1.1 & -1.2 \\ -1.3 & -1.4 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$$

Given A - $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} 3 & 1 \\ 0 & -1 \\ -2 & 3 \end{pmatrix}$

Find C = A × B

Solution:

Step 1: Multiply the elements in the first row of A with the corresponding elements in the first column of B. Add the products to get the element C_{11}

$$\begin{array}{r} \text{row 1} \longrightarrow \\ \\ \end{array}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{pmatrix}\underset{\substack{\uparrow \\ \text{column 1}}}{\begin{pmatrix} 3 & 1 \\ 0 & -1 \\ -2 & 3 \end{pmatrix}} = \begin{pmatrix} (1\times 3)+(2\times 0)+(-1\times -2) & \\ & \end{pmatrix} = \begin{pmatrix} 5 & \\ & \end{pmatrix}$$

step 1

$$\begin{pmatrix}1&2&-1\\2&0&1\end{pmatrix}\begin{pmatrix}3&1\\0&-1\\-2&3\end{pmatrix}=\begin{pmatrix}(1\times3)+(2\times0)+(-1\times-2) & \\ & \end{pmatrix}=\begin{pmatrix}5 & \\ & \end{pmatrix}$$

Step 2: Multiply the elements in the first row of A with the corresponding elements in the second column of B. Add the products to get the element C_{12}

$$\text{row 1}\longrightarrow\begin{pmatrix}1&2&-1\\2&0&1\end{pmatrix}\begin{array}{c}\begin{pmatrix}3&1\\0&-1\\-2&3\end{pmatrix}\\ \uparrow \\ \text{column 2}\end{array}=\begin{pmatrix}5 & (1\times1)+(2\times-1)+(-1\times3)\\ & \end{pmatrix}=\begin{pmatrix}5&-4\\ & \end{pmatrix}$$

Step 3: Multiply the elements in the second row of A with the corresponding elements in the first column of B. Add the products to get the element C_{21}

$$\text{row 2}\longrightarrow\begin{pmatrix}1&2&-1\\2&0&1\end{pmatrix}\begin{array}{c}\begin{pmatrix}3&1\\0&-1\\-2&3\end{pmatrix}\\ \uparrow \\ \text{column 1}\end{array}=\begin{pmatrix}5&-4\\(2\times3)+(0)+(1\times-2) & \end{pmatrix}=\begin{pmatrix}5&-4\\4& \end{pmatrix}$$

Step 4: Multiply the elements in the second row of A with the corresponding elements in the second column of B. Add the products to get the element C_{22}

$$\text{row 2}\longrightarrow\begin{pmatrix}1&2&-1\\2&0&1\end{pmatrix}\begin{array}{c}\begin{pmatrix}3&1\\0&-1\\-2&3\end{pmatrix}\\ \uparrow \\ \text{column 2}\end{array}=\begin{pmatrix}5&-4\\4&(2\times1)+(0\times-1)+(1\times3)\end{pmatrix}=\begin{pmatrix}5&-4\\4&5\end{pmatrix}$$

$$\text{So, } C=\begin{pmatrix}5&-4\\4&5\end{pmatrix}$$

Types of Matrices

Row Matrix
$\begin{pmatrix} a & b & c \end{pmatrix}$

Column Matrix
Vector Matrix
$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Zero Matrix
Null Matrix
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

Diagonal Matrix
$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

Scalar Matrix
$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$

Unit Matrix
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Upper Triangular Matrix
$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$

Lower Triangular Matrix
$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$

What are the types of matrices?

A matrix may be classified by types. It is possible for a matrix to belong to more than one type.

A **row matrix** is a matrix with only one row.
Example: E is a row matrix of order 1 × 1

$$E = \begin{pmatrix} 4 \end{pmatrix}$$

Example: B is a row matrix of order 1 × 3

$$B = \begin{pmatrix} 9 & -2 & 5 \end{pmatrix}$$

A **column matrix** is a matrix with only one column.
Example: C is a column matrix of order 1 × 1

$$C = \begin{pmatrix} 3 \end{pmatrix}$$

A column matrix of order 2 ×1 is also called a **vector** matrix.
Example: D is a column matrix of order 2 × 1

$$D = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$$

A **zero matrix** or a **null matrix** is a matrix that has all its elements zero.
Example: **O** is a zero matrix of order 2 × 3

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A **square matrix** is a matrix with an equal number of rows and columns.
Example: T is a square matrix of order 2 × 2

$$T = \begin{pmatrix} 6 & 3 \\ 0 & 4 \end{pmatrix}$$

Example: V is a square matrix of order 3 × 3

$$V = \begin{pmatrix} 7 & 1 & 9 \\ 3 & 2 & 5 \\ 2 & 1 & 8 \end{pmatrix}$$

A **diagonal matrix** is a square matrix that has all its elements zero except for those in the diagonal from top left to bottom right; which is known as the **leading diagonal** of the matrix. Example: B is a diagonal matrix.

$$B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

A **scalar matrix** is a diagonal matrix where all the diagonal elements are equal. For example:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

An **upper triangular matrix** is a square matrix where all the elements located below the diagonal are zeros. For example:

$$\begin{bmatrix} 2 & 3 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

A **lower triangular matrix** is a square matrix where all the elements located above the diagonal are zeros. For example:

$$\begin{bmatrix} 3 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 5 & 1 \end{bmatrix}$$

A **unit matrix** is a diagonal matrix whose elements in the diagonal are all ones. Example: P is a unit matrix.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Singular Matrix

A square matrix A is said to be singular if $|A| = 0$.

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \qquad |A| = 1 \times 8 - 4 \times 2 = 0$$

Non-Singular Matrix

A square matrix A is said to be non-singular if $|A| \neq 0$

Matrix

A matrix is a rectangular grid of numbers or symbols that is represented in a row and column format. Each individual term of a matrix is known as elements or entries. The matrix is determined with the number of rows and columns. For example, a matrix with 2 rows and 3 columns is referred to as a 2 x 3 matrix.

Matrix can also have an even number of rows and columns; these are known as square matrix.

A row vector is a matrix made up on only one row of numbers, while a column vector is a matrix that is made up of only one column of numbers.

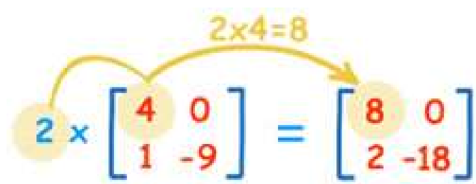
The matrices are usually enclosed in square or curved brackets. Each closed bracket is considered as a one matrix. These matrices are assigned a capital alphabet that represents the matrix.

The data in the matrix can be any type of number that we choose, including positive, negative, zero, fractions, decimals, symbols, alphabets, etc.

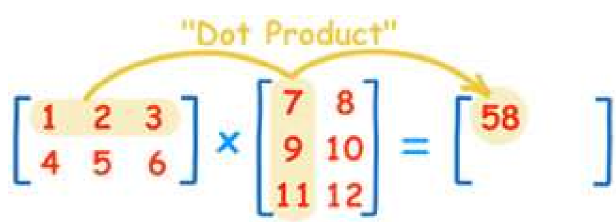
Matrices can be added, subtracted or multiplied. In case of addition, subtraction and multiplication of two matrices, the matrices must have the same number of rows and columns.

There are two forms of multiplication: scalar multiplication and multiplication of a matrix by another matrix.

Scalar matrix includes multiplying a matrix with a single number.



Multiplication of two matrices with each other requires solving them in a 'dot product', where a single row is multiplied with a single column. The resulting figures are then added up. The result of th first multiplication would be 1 x 7 + 2 x 9 + 3 x 11 = 58.



There are various different kinds of matrices: Square, diagonal and identity. A square matrix is a matrix that has the same number of rows and columns i.e.: 2x2, 3x3, 4x4, etc.

A diagonal matrix is a square matrix that has zeros as elements in all places, except in the diagonal line, which runs from top left to bottom right.

An identity matrix is a diagonal matrix that has all diagonal elements equal to 1.

Uses of Matrices:

Matrices are applied prominently in linear transformation, required for solving linear functions. Other fields that include matrices are classical mechanics, optics, electromagnetism, quantum mechanics, and quantum electrodynamics. It is also used in computer programming, graphics and other computing algorithms.

In computer graphics, they are used to project a 3-dimensional image onto a 2-dimensional screen. In probability theory and statistics, stochastic matrices are used to describe sets of probabilities.

Determinants

A determinant is a component of a square matrix and it cannot be found in any other type of matrix. A determinant is a real number that can be informally considered as the result of solving a square matrix. Determinant is denoted as det (matrix A) or |A|. It may seem like the absolute value of A, but in this case it refers to determinant of matrix A. The determinant of a square matrix is the product of the elements on the main diagonal minus the product of the elements off the main diagonal.

Let's assume the example of matrix B:

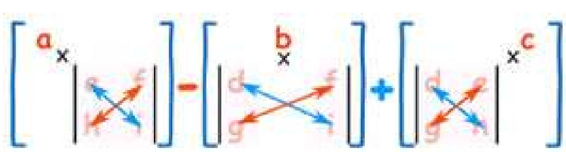
$$B = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

The determinant of matrix B or |B| would be 4 x 6 – 6 x3.

This would give the determinant as 6.

For a 3x3 matrix, a similar pattern would be used.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$



Properties of determinants:

1. The determinant is a real number, it is not a matrix.
2. The determinant can be a negative number.
3. It is not associated with absolute value at all except that they both use vertical lines.
4. The determinant only exists for square matrices (2×2, 3×3, ... n×n). The determinant of a 1×1 matrix is that single value in the determinant.
5. The inverse of a matrix will exist only if the determinant is not zero.

Difference between Matrix and a Determinant

1. Matrices do not have definite value, but determinants have definite value.
2. In a Matrix the number of rows and columns may be unequal, but in a Determinant the number of rows and columns must be equal.
3. The entries of a Matrix are listed within a large parenthesis (large braces), but in a determinant the entries are listed between two strips (i.e. between two vertical lines).
4. Let A be a Matrix. Matrix kA is obtained by multiplying all the entries of the Matrix by k. Let be any Determinant. is obtained by multiplying 'every entry of a row' or 'every entry of a column' by k

Difference between Matrix and Determinant:

A matrix or matrices is a rectangular grid of numbers or symbols that is represented in a row and column format. A determinant is a component of a square matrix and it cannot be found in any other type of matrix.

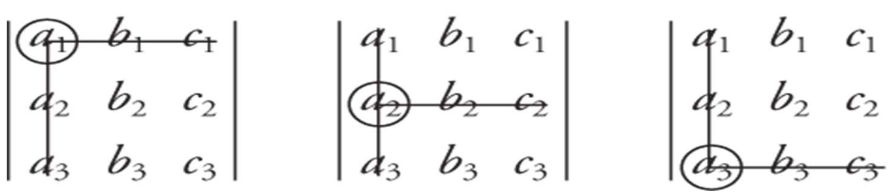
1. Matrices and determinants are important concepts in linear mathematics.
2. These concepts play a huge part in linear equations are also applicable to solving real-life problems in physics, mechanics, optics, etc.
3. A matrix is a grid of numbers, symbols or expressions that is arranged in a row and column format.
4. A determinant is a number that is associated with a square matrix.
5. These two terms can become quite confusing for people that are just learning these concepts.

The determinant of a 3 × 3 matrix can be defined as shown in the following.

minor determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \underset{\text{subtract}}{-} a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \underset{\text{add}}{+} a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Each minor determinant is obtained by crossing out the first column and one row.



Example 1

Evaluate the following determinant.

$$\begin{vmatrix} -2 & 4 & 1 \\ -3 & 6 & -2 \\ 4 & 0 & 5 \end{vmatrix}$$

First find the minor determinants.

$$\begin{vmatrix} -2 & 4 & 1 \\ -3 & 6 & -2 \\ 4 & 0 & 5 \end{vmatrix} \qquad \begin{vmatrix} -2 & 4 & 1 \\ -3 & 6 & -2 \\ 4 & 0 & 5 \end{vmatrix} \qquad \begin{vmatrix} -2 & 4 & 1 \\ -3 & 6 & -2 \\ 4 & 0 & 5 \end{vmatrix}$$

$$-2\begin{vmatrix} 6 & -2 \\ 0 & 5 \end{vmatrix} \quad \underset{\text{subtract}}{-} \quad -3\begin{vmatrix} 4 & 1 \\ 0 & 5 \end{vmatrix} \quad \underset{\text{add}}{+} \quad 4\begin{vmatrix} 4 & 1 \\ 6 & -2 \end{vmatrix}$$

$$-2(30-0) \quad - \quad -3(20-0) \quad + \quad 4(-8-6)$$

$$-60 \quad - \quad -60 \quad + \quad -56$$

$$-60 \quad\quad +60 \quad\quad -56 \quad = \quad -56$$

The solution is

$$\begin{vmatrix} -2 & 4 & 1 \\ -3 & 6 & -2 \\ 4 & 0 & 5 \end{vmatrix} = -56$$

Minors and Co-factors

It is to be noted that a1 is multiplied by a lower order determinant $\begin{vmatrix} b2 & c2 \\ b3 & c3 \end{vmatrix}$

Obtained by deleted the column and row containing a1, similarly, for the other terms, $\begin{vmatrix} b2 & c2 \\ b3 & c3 \end{vmatrix}$ is termed as minor of a1 in the original matrix. There will be as many minors as are the elements of the matrix.

Thus, the minor of b2 will be obtained by deleting the 2nd row and 2nd column,

In $\begin{vmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{vmatrix}$ minor of b2 $\begin{vmatrix} a1 & c1 \\ a3 & c3 \end{vmatrix}$

The cofactor of an element is the minor of that element multiplied by $(-1)^{i+j}$ when ith row and jth column have been deleted since they contain the element.

Now, a1 lies in the 1st row and 1st column, hence its cofactor is

$(-1)^{i+1} \begin{vmatrix} b2 & c2 \\ b3 & c3 \end{vmatrix}$ similarly b3 lies in 3rd row and 2nd column,

Hence its cofactor is $(-1)^{3+2} \begin{vmatrix} a1 & c1 \\ a2 & c2 \end{vmatrix}$

Example:

Find minor and cofactor of the 2 X 2 and 3 X 3 Determinant

$$\Delta = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$$

We have elements

$a_{11} = 3$
$a_{12} = 2$
$a_{21} = 1$
$a_{22} = 4$

Minor M_{11} = = $\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 4$

Minor M_{12} = = $\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 1$

Minor M_{21} = = $\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 2$

Minor M_{22} = = $\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 3$

Cofactor will be

$A_{11} = (-1)^{1+1} . M_{11} = (-1)^2 .4 = 4$

$A_{12} = (-1)^{1+2} . M_{12} = (-1)^3 .1 = -1$

$A_{21} = (-1)^{2+1} . M_{21} = (-1)^3 .2 = -2$

$A_{22} = (-1)^{2+2} . M_{22} = (-1)^4 .3 = 3$

Solving Simultaneous Equations

Cramer's Rule

To use determinants to solve a system of three equations with three variables (Cramer's Rule), say *x*, *y*, and *z*, four determinants must be formed following this procedure:

1. Write all equations in standard form.
2. Create the denominator determinant, *D*, by using the coefficients of *x*, *y*, and *z* from the equations and evaluate it.
3. Create the *x*-numerator determinant, D_x, the *y*-numerator determinant, D_y, and the *z*-numerator determinant, D_z, by replacing the respective *x*, *y*, and *z* coefficients with the constants from the equations in standard form and evaluate each determinant.

The answers for *x*, *y*, and *z* are as follows: $x = \frac{D_x}{D}, \; y = \frac{D_y}{D}, \; z = \frac{D_z}{D}$

Example

Solve this system of equations, using Cramer's Rule.

$$\begin{cases} 3x + 2y - z = 2 \\ 2x - y - 3z = 13 \\ x + 3y - 2z = 1 \end{cases}$$

Find the minor determinants.

$$\begin{array}{cccc} & x\text{-coefficients} & y\text{-coefficients} & z\text{-coefficients} \\ & \downarrow & \downarrow & \downarrow \\ D = & \begin{vmatrix} 3 \\ 2 \\ 1 \end{vmatrix}\!\!\!\! & \begin{matrix} 2 \\ -1 \\ 3 \end{matrix} & \begin{matrix} -1 \\ -3 \\ -2 \end{matrix} \end{array}$$

$$D = \begin{vmatrix} 3 & 2 & -1 \\ 2 & -1 & -3 \\ 1 & 3 & -2 \end{vmatrix} = 3\begin{vmatrix} -1 & -3 \\ 3 & -2 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ -1 & -3 \end{vmatrix}$$

$$= 3[2-(-9)] -2[-4-(-3)] +1(-6-1)$$

$$= 3(11) - 2(-1) + 1(-7)$$

$$= 33 + 2 - 7 = 28$$

Use the constants to replace the *x*-coefficients.

constants replacing the *x*-coefficients

$$D_x = \begin{vmatrix} 2 & 2 & -1 \\ 13 & -1 & -3 \\ 1 & 3 & -2 \end{vmatrix} = 2\begin{vmatrix} -1 & -3 \\ 3 & -2 \end{vmatrix} - 13\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ -1 & -3 \end{vmatrix}$$

$$= 2[2 -(-9) -13[-4 -(-3)] +1(-6 -1)$$

$$= 2(11) - 13(-1) + 1(-7)$$

$$= 22 + 13 - 7 = 28$$

Use the constants to replace the *y*-coefficients.

constants replacing the *y*-coefficients

$$D_y = \begin{vmatrix} 3 & 2 & -1 \\ 2 & 13 & -3 \\ 1 & 1 & -2 \end{vmatrix} = 3\begin{vmatrix} 13 & -3 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 13 & -3 \end{vmatrix}$$

$$= 3[-26-(-3)] -2[-4 -(-1)] +1[-6-(-13)]$$

$$= 3(-23) - 2(-3) + 1(7)$$

$$= -69 + 6 + 7 = -56$$

Use the constants to replace the *z*-coefficients.

constants replacing the z-coefficients

$$D_z = \begin{vmatrix} 3 & 2 & 2 \\ 2 & -1 & 13 \\ 1 & 3 & 1 \end{vmatrix} = 3\begin{vmatrix} -1 & 13 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ -1 & 13 \end{vmatrix}$$

$$= 3(-1-39) - 2(2-6) + 1[26-(-2)]$$

$$= 3(-40) - 2(-4) + 1(28)$$

$$= -120 + 8 + 28 = -84$$

Therefore,

$$x = \frac{D_x}{D} = \frac{28}{28} = 1,\quad y = \frac{D_y}{D} = -\frac{56}{28} = -2,\quad z = \frac{D_z}{D} = -\frac{84}{28} = -3$$

The check is left to you. The solution is $x = 1$, $y = -2$, $z = -3$.

If the denominator determinant, D, has a value of zero, then the system is either inconsistent or dependent. The system is dependent if all the determinants have a value of zero. The system is inconsistent if at least one of the determinants, D_x, D_y, or D_z, has a value not equal to zero and the denominator determinant has a value of zero.

Cramer's Rule: The method of determinants

A system of two equations in two unknowns has this form:

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

The *a*'s are the coefficients of the *x*'s. The *b*'s are the coefficients of the *y*'s. The following is the matrix of those coefficients.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

The *number* $a_1b_2 - b_1a_2$ is called the determinant of that matrix.

$$\det \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$$

Let us denote that determinant by D.

Now consider this matrix in which the *c*'s replace the coefficients of the *x*'s:

$$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

Then the determinant of that matrix -- which we will call D*x* is $c_1b_2 - b_1c_2$

And consider this matrix in which the *c*'s replace the coeffients of the *y*'s:

$$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The determinant of that matrix – D*y* is $a_1c_2 - c_1a_2$

Cramer's Rule then states the following:

In every system of two equations in two unknowns
in which the determinant D is not 0,

$$x = \frac{Dx}{D}$$

$$y = \frac{Dy}{D}$$

Use Cramer's Rule to solve this system of equations:

$5x+3y=-11$
$2x+4y=-10$

Solution.

$$D=\det\begin{vmatrix}5 & 3\\ 2 & 4\end{vmatrix}=5\cdot 4-3\cdot 2$$

$$=20-6=14$$

$$Dx=\det\begin{vmatrix}-11 & 3\\ -10 & 4\end{vmatrix}=-11\cdot 4-3\cdot -10$$

$$=-44+30=-14$$

$$Dy=\det\begin{vmatrix}5 & -11\\ 2 & -10\end{vmatrix}=5\cdot -10-(-11)\cdot 2$$

$$=-50+22$$

$$=-28.$$

Therefore,

$$x=\frac{Dx}{D}=\frac{-14}{14}=-1.$$

$$y=\frac{Dy}{D}=\frac{-28}{14}=-2.$$

Use Cramer's Rule to solve these simultaneous equations.

$3x-5y=-31$
$2x+\ y=1$

$$D=\ \det\begin{vmatrix}3 & -5\\ 2 & 1\end{vmatrix}=\ 3\cdot 1-(-5)\cdot 2$$

$$=\ 3+10=13$$

$$Dx=\ \det\begin{vmatrix}-31 & -5\\ 1 & 1\end{vmatrix}=\ -31\cdot 1-(-5)\cdot 1$$

$$=\ -31+5=-26$$

$$Dy=\ \det\begin{vmatrix}3 & -31\\ 2 & 1\end{vmatrix}=\ 3\cdot 1-(-31)\cdot 2$$

$$=\ 3+62=65$$

$$x=\frac{Dx}{D}=\frac{-26}{13}=-2.$$

$$y=\frac{Dy}{D}=\frac{65}{13}=5.$$

Advantages and Disadvantages of Cramer's Rule

Advantages

1) Cramer's Rule is that you can find the value of x, y, or z without having to know any of the other values of x, y, or z.

For example, if you needed to find just the value of y, Cramer's Rule would work well.

2) Cramer's Rule is that if any of the values of x, y, or z is fractions, you do not have to plug in a fraction to find the other values. Each value can be found independently.

Disadvantages

One of the only disadvantages to using Cramer's rule is if the value of D is zero then Cramer's Rule will not work because you cannot divide by zero. However, if the value of D is zero then you know that the solution is either "No Solution" or "Infinite Solutions". You will have to use a different technique such as Addition/Elimination to find out whether the answer is "No Solution" or "Infinite Solutions.

UNIT –V

Input-Output Analysis
Definition

Input-output analysis is a technique which was invented by Prof. Wassily W. Leontief in the year 1951. This technique deals with the type of problems, one of which may be described in the following words:

"What should be the level of output of each industry with the existing technology so that the total output goal for consumer and industrial use of its product gets fully satisfied; or alternatively, what level of output of each producing sector in an economy can bring about equilibrium for its product in the economy as whole."

The basic idea behind this problem is quite simple to understand. Since Inputs of one industry are the outputs of another industry and vice versa, ultimately their mutual relationship must lead to equilibrium between supply and demand in the economy consisting of n industries.

Thus, the essence of input-output analysis is that, given certain technological coefficients and final demand, each endogenous sector would find its output uniquely determined as a linear combination of multi-sector demand. Let us suppose that an economic system consists of n producing sectors.

In order to avoid bottlenecks in the economy, the total output of each producing sector must satisfy the total demand for its product which, in fact, would arise because:

1. Its product is being used as an intermediate product (i.e., input) elsewhere in the industrial structure of production, and
2. Its product is used for household consumption, capital formation, Government consumption or for export.

For example, total output of agricultural sector may be used (or demanded):

1. as an input in food or other manufacturing sector (grain for producing bread or cotton for producing cloth); and
2. as a final consumption by Government or households (vegetables or grain) and/or as an export demand.

Let us assume that an economy consists of 4 producing sectors only; and that the production of each sector is being used as an input in all the sectors and is used for final consumption.

Suppose:

1. X1, X2, X3, and X4, are total outputs of the 4 sectors;
2. F1, F2, F3, and F4, are the amounts of final demand, consumption, Capital formation and exports for output of these sectors.

We can now translate the distribution of total product of 4 producing sectors in the following way:

Input-Output Transaction Table

Producing sector No.	**Total output of the sector**	**Input requirements of Producing sectors**				**Requirements for final uses**
		X_1	X_2	X_3	X_4	
1	2	3	4	5	6	7
1	X_1	X_{11}	X_{12}	X_{13}	X_{14}	F_1
2	X_2	X_{21}	X_{22}	X_{23}	X_{24}	F_2
3	X_3	X_{31}	X_{32}	X_{33}	X_{34}	F_3
4	X_4	X_{41}	X_{42}	X_{43}	X_{44}	F_4
Primary Input (Labour)	Total Primary Input =L→	L1	L2	L3	L4	--

We derive two important equations from the above table:

(1) Columns 3, 4, 5 and 6 of the above table give us total inputs (from all sectors) utilised by each sector for its production. In other words, col. 3 gives the production function of sector 1 and col.6 represents the production function of sector 4.

$X_1 = f_1(X_{11}, X_{21}, X_{31}, X_{41}, L_1)$
$X_2 = f_1(X_{12}, X_{22}, X_{32}, X_{42}, L_2)$
$X_3 = f_1(X_{13}, X_{23}, X_{33}, X_{43}, L_3)$
$X_4 = f_1(X_{14}, X_{24}, X_{34}, X_{44}, L_4)$

In General terms, if there are n number of producing sectors then production function of sector n will be represented by $Xn = fn(X_{1n}, X_{2n}, X_{3n}, X_{4n}, Ln)$

(2) Rows of the table give us the equality between the demand and supply of each product:

$X_1 = X_{11} + X_{21} + X_{31} + X_{41} + F_1$

$X_2 = X_{21} + X_{22} + X_{23} + X_{24} + F_2$

$X_2 = X_{31} + X_{32} + X_{33} + X_{34} + F_3$

$X_2 = X_{41} + X_{42} + X_{43} + X_{44} + F_4$

$L = L_1 + L_2 + L_3 + L_4$

That is , $X_i = \sum_{i=1}^{n} xij + F_i$ and $L = \prod_{i=1}^{n} L_i$

X=X4+X42X2 +X+F

Here, X_i = Total output of ith sector

X_{ij} = Output of ith sector used as input in jth sector, and

F_i= Final demand for ith sector.

The above identity states that all the output of a particular sector could be utilised either as an input in one of the producing sectors of the economy and/or as a final demand. Basically, therefore, input-output analysis is nothing more than finding the solution of these simultaneous equations.

Assumptions

The economy can be meaning fully divided into a finite number of sectors (industries):

1. Each industry produces only one homogeneous output. No two products are produced jointly; but if at all there is such a case then it is assumed that they (products) are produced in fixed proportions.
2. Each producing sector satisfies the properties of linear homogeneous production function--in other words, production of each sector is subject to constant returns to scale so that k-fold change in every input will result in an exactly k-fold change in the output.
3. A far stronger assumption is that each industry uses a fixed input ratio for the production of its output; in other words, Input requirements per unit of output in each sector remain fixed and constant. The level of output in each sector (Industry) uniquely determines the quantity of each input which is purchased.

Closed and Open Input-Output Models

our model contains exogenous sector of final demand which supplies primary input factors (labour services - which are not produced by n industries) and consumes the outputs of the n-producing industries (not as input). Such an input-output model is known as open model.

It includes exogenous sectors in terms of "final demand bill" along with the endogenous sectors in terms of n-producing sectors. Input-output model which has endogenous final demand vector is known as closed input- output model.

Coefficient Matrix and Open Model

Our open model in matrix notation is given by:

$$X = AX+F$$

Where A is the input coefficient matrix, F is the final demand vector and X is the total output matrix.

The input coefficient matrix represented by [a_{ij}]

$$\begin{bmatrix} a11 & a12 & a13 & a14 \\ a21 & a22 & a23 & a2n \\ \dots\dots & & \dots.. & \dots.. \\ an1 & an2 & an3 & amn \end{bmatrix}$$

is of great importance. Each column of this matrix specifies the input requirements for the production of one unit of a particular commodity. The second column, for example, states that to produce a unit of commodity 2, the inputs needed are a_{12} units of commodity 1, a_{21}, units of commodity 2, a_{32} units of commodity 3, and a_{n2} units of commodity n.

$$\begin{bmatrix} 0 & a12 & a13 & a14 \\ a21 & 0 & a23 & a2n \\ \dots\dots & & \dots.. & \dots.. \\ an1 & an2 & an3 & 0 \end{bmatrix}$$

If no industry uses its own product as an input then,

(Note. Elements of matrix will be zero whenever the sectors do not trade with each other, It should be noted that no coefficient can be negative, there can be no negative inputs.)

Coefficient Matrix and Closed Model

We shall now examine whether we will be able to estimate F or X if the model is changed into closed one. If the exogenous sector (final demand bill) of the open input-output model is absorbed into the system of endogenous sectors, the model would turn into a closed one. In such a model final demand bill and primary inputs will not appear anymore; rather in their place, we shall have the input requirements and output of this newly conceived industry, the 'household industry' producing the primary input labour. Final demand sector would now be considered as one of endogenous sectors. As such now we shall have (n + 1) industries in place of n industries and all producing for the sake of satisfying the input requirements.

This newly conceived industry (of final demand bill) will also be assumed to have a fixed input ratio as any other industry. In other words, the supply of primary input must now bear a fixed proportion to final demand (i.e., consumption of this newly conceived industry). This will mean, for example, that households will consume each commodity in fixed proportion to the labour services they supply.

Looking at the problem in this particular way, it appears that the conversion of open model into a closed one should not create any significant change in our analysis and solution because disappearance of final demand means only an addition of one more homogeneous equation to already existing set of n homogeneous equations. Is it really so? Let us examine.

Let us assume that there are four industries only - including the new one (of final demand) designated by subscript 0. We shall, therefore, have the following set of equations:

This gives us a homogeneous equation system

$$X_0 = a_{00}X_0 + a_{01}X_1 + a_{02}X_2 + a_{03}X_3$$

$$X_1 = a_{10}X_0 + a_{11}X_1 + a_{12}X_2 + a_{13}X_3$$

$$X_2 = a_{20}X_0 + a_{21}X_1 + a_{22}X_2 + a_{23}X_3$$

$$X_3 = a_{30}X_0 + a_{31}X_1 + a_{32}X_2 + a_{33}X_3$$

This gives us a homogeneous equation system

$$\begin{bmatrix} (1-a_{00}) & -a_{01} & -a_{02} & -a_{03} \\ -a_{10} & (1-a_{11}) & -a_{12} & -a_{13} \\ -a_{20} & -a_{21} & (1-a_{22}) & -a_{23} \\ -a_{30} & -a_{31} & -a_{32} & (1-a_{13}) \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[i - A] = X = 0$$

Since the 4 rows of the input coefficient matrix happen to be linearly dependent, I - A will turn out to be zero. Hence the solution is indeterminate.

This means that in a closed model no unique output-mix of each sector exists. We can at most determine the output levels of endogenous sectors in proportion to one another, but cannot fix their absolute levels unless additional information is made available exogenously.

Uses of Input –Output Analysis

1. A producer can know from the input-output table, the varieties and quantities of goods which he and the other firms buy and sell to each other.
2. It is also possible to find out from the input-output table the interrelations among firms and industries about possible trends towards combinations.

3. The input-output model has come to be used for national income accounting because it provides a more detailed breakdown of the macro aggregates and money flows.
4. It provides for individual branches of the economy's estimates of production and import levels that are consistent with each other and with the estimates of final demand.
5. The input-output model aids in the allocation of the investment required to achieve the production levels in production programme.
6. The analysis of import requirements and substitution possibilities is facilitated by the knowledge of the use of domestic and import materials in different branches of an industry. In addition to direct requirements of capital, labour and imports, the indirect requirements in other sectors of an industry can also be estimated.

Limitation of Input-Output Analysis

1. Errors in forecasting final demand will have grave consequences
2. Current relative prices of inputs may not be same as the ones implied in the table
3. The assumption of linear homogenous production function may not be valid. The technical coefficients will not remain constant even if input price ratios are held constant in such circumstances
4. The constant coefficient formulation also ignores the possibility of industry outputs reaching capacity, changing prices and input proportions in the table
5. The assumption of constant technical coefficients goes counter to the possibility of substitution of inputs and factors
6. Sect oral division is for practical purposes, limited. Such a Sectorisation is not good enough for many forecasting purposes.
7. Sectorisation (grouping of commodities in sectors) is often arbitrary the intro-sectoral heterogeneity with respect to technologies, efficiency and demand is not invariant over time.
8. Input output model building is highly costly in terms of time and money
9. Regional input-output analysis involves many more assumptions and difficulties in construction of such tables.